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2009 J. Phys. A: Math. Theor. 42 205202

(http://iopscience.iop.org/1751-8121/42/20/205202)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 205202 (7pp) doi:10.1088/1751-8113/42/20/205202

The closed-form solution for the 2D Poisson equation with a rectangular boundary

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Received 17 September 2008, in final form 3 April 2009
Published 29 April 2009
Online at stacks.iop.org/JPhysA/42/205202

Abstract
The 2D Green's function for the Poisson equation with a rectangular boundary is investigated using the Schwarz–Christoffel transformation method. A closed form of the Green's function containing Jacobi elliptic functions is developed.

PACS numbers: 42.65.Tg, 02.30.Fn

1. Introduction

In many areas of physics, such as electrostatics [1, 2], thermodynamics [3, 4], optics [3–5], hydrodynamics [6] and elasticity [7, 8], problems are often formulated using Poisson equations. As a typical problem, consider laser-induced thermal self-focusing [3–5]. Optical energy is weakly absorbed with an absorption coefficient α and diffused with a thermal conductivity κ, yielding a temperature gradient whose distribution is governed by the Poisson equation. Specifically

$$\begin{aligned} &\nabla^2 T(x, y) = -\frac{\alpha}{\kappa} I(x, y), \\ &T(x, y)|_{\text{boundary}} = 0, \end{aligned} \tag{1}$$

where $T(x, y)$ is the temperature distribution, $I(x, y) = |A(x, y)|^2$ is the light intensity of the paraxial optical beam $A(x, y)$, and a homogeneous boundary condition is used for simplicity. Equation (1) together with the equation for a paraxial optical beam $A(x, y)$ are used to model the propagation of an optical beam in lead glass [4], a system of nonlocal nonlinear media. Under some conditions and with rectangular boundary conditions this system supports spatial optical solitons. Although significant experimental progress has been made [3–5], analytic studies that elucidate the underlying physics are rare.

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Analytical solutions of the Poisson equation can be obtained using methods such as separation of variables [9, 10], eigenfunction expansion [11], image methods, etc. With complicated boundary conditions, however, the Poisson equation may well be unsolvable. In particular, a closed-form solution of the Poisson equation with rectangular boundary conditions is, to our knowledge, an open problem, although an infinite series solution can be deduced from eigenfunction expansion or image methods [12] and several numerical methods are promising [13]. Conformal transformation is a powerful technique to convert, through an analytical transformation, an intricate boundary problem in the z-plane to a trivial one in the w-plane. The conformal Schwarz–Christoffel transformation (SCT) is a recipe for converting a polygon boundary and thus may be used to solve the Poisson equation with rectangular boundary conditions.

In this paper we investigate the 2D Poisson equation with rectangular boundary conditions using a method based on the SCT. We develop a closed-form Green's function that describes the thermal response of a point source of light in lead glass. This is an important step forward in the analytical study of laser-induced thermal spatial solitons in lead glass.

2. Invariability of the Poisson equation under the SCT

We now solve the primary problem formulated by equation (1) using the SCT method. It is known that under conformal transformations $w = f(z)$ (where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ are points in the z-plane and points in the w-plane, respectively) the Poisson equation (1) is formally invariant [14]. Explicitly,

$$\nabla^2 T(u, v) = -\frac{\alpha}{\kappa}\bar{I}(u, v), \tag{2}$$

where $\bar{I}(u, v) = |f'(z)|^2 I(x, y)$, implying that the light intensity expands by a factor of $|f'(z)|^2$. The solution of the Poisson equation using a Green's function is given by

$$T(x, y) = -\frac{\alpha}{\kappa}\int I(x_0, y_0)G(x, y, x_0, y_0)\,\mathrm{d}x_0\,\mathrm{d}y_0, \tag{3}$$

where $G(x, y, x_0, y_0)$ is the Green's function (also called the response function [15]) representing the thermal material response to a point source of light $\delta(x_0, y_0)$. The integration is performed over an area within a given boundary that is exposed to light intensity.

We concentrate on the Green's function through which the problem may be easily solved by integration (3). The Green's function satisfies the equation

$$\begin{aligned} &\nabla^2 G(x, y, x_0, y_0) = -\delta(x - x_0, y - y_0);\\ &G(x, y, x_0, y_0)|_{\text{boundary}} = 0; \end{aligned} \tag{4}$$

which under a conformal transformation will be converted to the form

$$\nabla^2 G(u, v, u_0, v_0) = |f^{'}(z)|^2\delta(x - x_0, y - y_0). \tag{5}$$

As required to treat the problem in the w-plane, in which the boundary condition is usually simplified, we rewrite the Dirac function as

$$\delta(x - x_0, y - y_0) = \frac{\delta(u - u_0, v - v_0)}{|J|}, \tag{6}$$

where

$$|J| = \frac{D(x, y)}{D(u, v)} = \frac{1}{\frac{D(u,v)}{D(x,y)}} = \frac{1}{\begin{vmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial v/\partial y\end{vmatrix}} \tag{7}$$

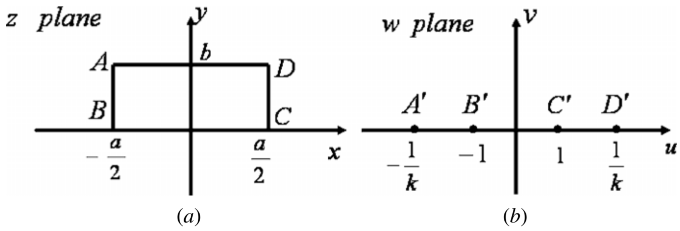

Figure 1. (*a*) The scheme of the primary electrostatics problem in the z-plane. The dimensions of the rectangle are $a \times b$. (*b*) The post-conversion points on the real axis in the w-plane correspond in order to the vertices of the rectangle in the z-plane.

is the Jabobi determinant. However, according to the Cauchy–Riemann equations the magnification factor in equation (5) is

$$|f'(z)|^2 = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{D(u, v)}{D(x, y)}. \tag{8}$$

We thus find the novel Green's function equation describing the variables in the w-plane

$$\begin{aligned} &\nabla^2 G(u, v, u_0, v_0) = -\delta(u - u_0, v - v_0), \\ &G(u, v, u_0, v_0)|_{\text{boundary}} = 0, \end{aligned} \tag{9}$$

which is of the same form as equation (4) but with simplified boundary conditions.

3. The closed form Green's function for the Poisson equation with rectangular boundary conditions

The essential work of the SCT is mapping the rectangular boundary of the original problem in the z-plane to the real axis in the w-plane with point-to-point correspondence. Figure 1 visualizes the counterpoints in the two complex planes. Figure 1(*a*) denotes the rectangular boundary of the primary problem in the z-plane. The dimensions of the rectangular boundary are $a \times b$. The SCT maps the interior of the rectangle in the z-plane to the upper half of the w-plane and converts the vertices of the rectangle A, B, C, D to the points A', B', C', D' on the real axis of the w-plane pictured in figure 1(*b*). The transform equation is

$$\begin{aligned} z &= Sk \int \frac{1}{\sqrt{1 - w^2}\sqrt{1 - k^2 w^2}}\, \mathrm{d}w \\ &= S' K(w, k) + C, \end{aligned} \tag{10}$$

where $S' = Sk$ and C are the suitably chosen integration constants determined by the size and position of the rectangle, and $K(w, k)$ is the elliptic integral of the first kind with the module k.

We note that $w = f(z = 0) = 0$ indicates that the origins of the two planes are counterpoints determined by the position of the rectangle. Furthermore, $w = f(z = a/2) = 1$ and $w = f(z = a/2 + \mathrm{i}b) = 1/k$ reflect the size of the rectangle since the rectangle is unique only when the lengths of two adjacent sides are given. Consequently the constants in equation (10) are $C = 0$, $S' = a/(2K)$, where K is the complete elliptic integral of the first kind. The ratio of the sides of the rectangle is $a/b = 2K/K'$, and results from the relation $K(1/k, k) = K + \mathrm{i}K'$ in which K' is the complete elliptic integral of the first kind with the residual module k' obeying $k^2 + k'^2 = 1$. Therefore both the module k and the residual module k' can be derived when a/b is given.

Finally we arrive at the transform equation mapping each point in the w-plane to a point in the z-plane.

$$z = \frac{a}{2K} K(w, k). \tag{11}$$

The inverse function of equation (11) is

$$w = \mathrm{sn}\left(\frac{2K}{a} z, k\right), \tag{12}$$

where sn is the Jacobi elliptic sine function. The mapping relations between the real and imaginary parts of points w and z are

$$\begin{aligned} u &= \frac{\mathrm{sn}(2Kx/a, k)\,\mathrm{dn}(2Ky/a, k')}{1 - \mathrm{dn}^2(2Kx/a, k)\,\mathrm{sn}^2(2Ky/a, k')}, \\ v &= \frac{\mathrm{cn}(2Kx/a, k)\,\mathrm{dn}(2Kx/a, k)\,\mathrm{sn}(2Ky/a, k')\,\mathrm{cn}(2Ky/a, k')}{1 - \mathrm{dn}^2(2Kx/a, k)\,\mathrm{sn}^2(2Ky/a, k')}, \end{aligned} \tag{13}$$

where cn, dn are the Jacobi elliptic cosine functions with module k or residual module k'.

We now rewrite the Poisson equation with the boundary condition in the w-plane.

$$\begin{aligned} &\nabla^2 G(u, v, u_0, v_0) = -\delta(u - u_0, v - v_0), \\ &G(u, v, u_0, v_0)|_{v=0} = 0. \end{aligned} \tag{14}$$

The solution of equation (14), obtained using the image method, is

$$G(u, v, u_0, v_0) = \frac{1}{4\pi} \ln \frac{(u - u_0)^2 + (v + v_0)^2}{(u - u_0)^2 + (v - v_0)^2}. \tag{15}$$

Equation (15) describes the sum potentials of a positive point charge at (u_0, v_0) and a negative point charge at $(u_0, -v_0)$. Using the mapping relations given by equations (13), equation (15) gives the solution to the primary problem formulated by equation (4).

The solids lines in figures 2(*a*)–(*e*) show the Green's function in the z-plane deduced using the SCT method. The scaling is normalized by the width a of the rectangle. The variable n is the ratio of the widths which is given by $n = a/b$, p and q are the normalized coordinates of the point source of light and are given by $p = x_0/(a/2)$ and $q = (y_0 - b/2)/(b/2)$, respectively. Figures 2(*a*) and (*b*) are the counter maps of the Green's function with the same $n = 2$, but with $p = 0.4, q = 0.4$ and $p = 0.1, q = 0.1$, respectively. Figure 2(*a*) clearly reflects the distortion of the Green's function profile due to the influence of the asymmetric boundary condition for the case of off-center excitation. In figure 2(*b*), reducing the displacement of the point source of light (i.e., x_0) while keeping the boundary scaling fixed is equivalent to increasing the rectangular scaling (i.e., a) while keeping the source fixed. Thus the curve is symmetric for the case of 'large boundary scaling' in spite of the off-center excitation, which implies an absence of the boundary influence. Figures 2(*c*) and (*d*) are the $y = y_0$ and $y = 0.6y_0$ section graphs of figure 2(*a*), respectively. An infinity denoting the infinity of the exciting density appears in figure 2(*c*) at the point source (x_0, y_0), indicating that (x_0, y_0) is a singular point. However, a finite maximum departing slightly from the source survives in figure 2(*d*). We believe that the more the section departs from $y = y_0$ the closer the finite maximum of the curve approaches the lateral center, but the weaker the Green's function becomes. Figure 2(*e*) also shows the $y = 0.6y_0$ section graphs based on the same parameters as for figure 2(*d*) except that $n = 10$. In figure 2(*e*), as b becomes narrower so does the full width at half maximum (FWHM) of the Green's function in the x-direction, which indicates the mutual influence of the x and y dimensions. The FWHM of the Green's function is important in that it is the decisive parameter for the nonlocality of the material in the study of the nonlocal

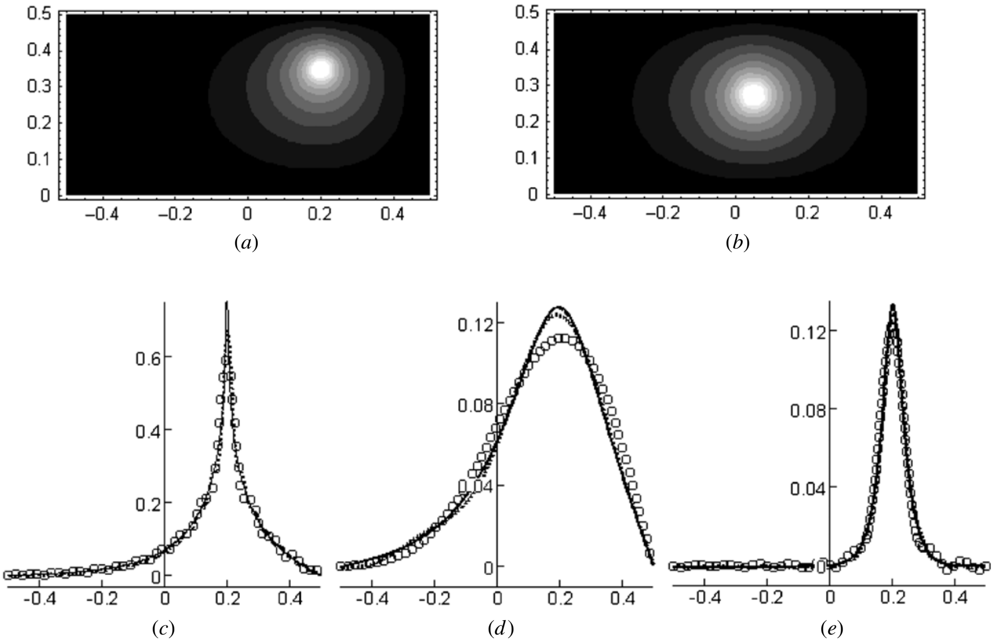

Figure 2. The Green's function varying with variables in the z-plane. The scaling is normalized by the side width a of the rectangle. (a) and (b) are the counter maps of the Green's function with the same $n = 2$ but $p = 0.4, q = 0.4$ and $p = 0.1, q = 0.1$ respectively where n, p and q in order denote the ratio of the side widths, the normalized coordinates of the point source of light formulated as $b = a/n$, $p = x_0/(a/2)$ and $q = (y_0 - b/2)/(b/2)$. (c) and (d) are the $y = y_0$ and $y = 0.6y_0$ section graphs of (a), respectively. (e) is also the $y = 0.6y_0$ section graph with the same parameters as (d) except $n = 10$. Solid line: the Green's function deduced by the conformal transformation method, dashed line: the Green's function deduced by the eigenfunction expansion method with $m = 50, m = 5$ and $m = 30$ for (c)–(e), respectively, open circles: the Green's function deduced by the eigenfunction expansion method with $m = 30, m = 3$ and $m = 20$ for (c)–(e), respectively.

spatial solitons [16]. Our result gives a useful suggestion to get a designed FWHM of the response function in one dimension by adjusting the scale of the other dimension.

We now compare the closed-form solution and the series solution based on the eigenfunction expansion method [14]

$$G(x, y, x_0, y_0) = \sum_{n=1}^{m} Gn(y) \sin\frac{n\pi}{a}\left(x + \frac{a}{2}\right), \tag{16}$$

is the exact solution in the case of $m = \infty$, where $\sin\frac{n\pi}{a}\left(x + \frac{a}{2}\right)$ is the nth order eigenfunction. The corresponding coefficient $Gn(y)$ is

$$\begin{aligned} Gn(y) = \frac{1}{n\pi(f(b)-1)}\bigg(&\frac{1}{f(y_0)f(y)}\sin\frac{n\pi(a+2x_0)}{2a}(f(b)-f(y_0))(f(y)-1)U(b-y_0) \\ &- (f(b)-f(y))(f(y_0)-1)U(-y_0) + (f(y_0)-f(y))(f(b)-1)U(y-y_0)\bigg), \end{aligned} \tag{17}$$

where $f(q) = \exp(n\pi q/a)$ and U is the unit step function. In figure 2 the dashed lines describe the Green's functions deduced using the eigenfunction expansion method with $m = 50, m = 5$

and $m = 30$ for figures 2(*c*)–(*e*), respectively. The open circles are the Green's functions deduced using the eigenfunction expansion method with $m = 30, m = 3$ and $m = 20$ for figures 2(*c*)–(*e*), respectively. We note that the convergence rate of the series solution alters with a/b and with position. In general the convergence rates are slow especially in the case of the solution with a singular point (figure 2(*c*)). Even for the case of the fastest convergence rate (figure 2(*d*)) the series solutions with $m = 5$ (dashed line) and the closed form solution are readily distinguished.

4. Conclusions

We investigated an analytical solution of the Poisson equation in the rectangular domain for a typical problem of laser-induced thermal self-focusing in lead glass. By means of the SCT technique a closed form of Green's function is deduced which is of great merit compared with the series solution. The asymmetry of the solution arises from the asymmetry of the boundary conditions in the case of off-center excitation but is invalid for sufficiently large boundary scaling. The FWHM of the solutions in two orthogonal dimensions interact, suggesting a method of response function design. The characteristics of the response function are critical factors that allow us to make qualitative conclusions about soliton formation and propagation behavior in media. These findings are therefore an important step forward for the analytical study of laser-induced thermal spatial solitons.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (grants nos 10474023 and 10674050), Specialized Research Fund for the Doctoral Program of Higher Education (grant no 20060574006), and Program for Innovative Research Team of the Higher Education in Guangdong (grant no 06CXTD005).

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